June 2

<u>Galois's criterion</u> Let K be char 0 field
Let $f \in K[x]$ be a polynomial
Let L be splitting field of f.
Then $f(x)$ is solvable via radicals $\iff$ $Gal(L/K)$ solvable.

We will prove: $\implies$

<u>Spectacular corollary</u> $\exists$ degree 5 $f(x) \in \mathbb{Q}[x]$
which are not solvable by radicals.

<u>Reason:</u> ① $\exists f \in \mathbb{Q}[x]$ s.t. $Gal(L/\mathbb{Q}) \cong S_5$

② S_5 is not solvable

Lemma 1 Let K be char 0 field.
Let ζ be a prim n^{th} root of unity.
Then $K(\zeta)$ Galois ext of K &
$\mathrm{Gal}(K(\zeta)/K)$ abelian.

Lemma 2 Let K char 0 field.
Assume K contains a primitive n^{th} root of unity.
Let α be a root of $x^n - a \in K[x]$.
Then $K(\alpha)$ is Galois$/K$ &
$\mathrm{Gal}(K(\alpha)/K)$ abelian.

Ex: Let $\zeta = e^{2\pi i/p}$ prim p^{th} rt.

Let $\alpha = \sqrt[p]{2}$

$$\mathbb{Q} \subset \mathbb{Q}(\zeta) \subset \mathbb{Q}(\zeta, \alpha)$$

Galois$/\mathbb{Q}$

$\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/p)^\times = \mathbb{Z}/(p-1)$

Galois$/\mathbb{Q}(\zeta)$ w/ $\mathrm{Gal} \cong \mathbb{Z}/p$

HW $\exists$
$$\mathrm{Gal}(\mathbb{Q}(\zeta,\alpha)/\mathbb{Q}) \cong \mathbb{Z}/p \rtimes (\mathbb{Z}/p)^\times$$

Fund thm $\exists$

$$\mathbb{Z}/p \trianglelefteq \mathbb{Z}/p \rtimes (\mathbb{Z}/p)^\times$$
$$\mathrm{Gal}(\mathbb{Q}(\zeta,\alpha)/\mathbb{Q}(\zeta)) \subset \mathrm{Gal}(\mathbb{Q}(\zeta,\alpha)/\mathbb{Q})$$

w/ quotient $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/p)^\times$

Lemma 3 Consider field exts

$$K \subset E \subset L.$$

Assume
- E normal/K
- L is radical & normal/K

$\Longrightarrow Gal(E/K)$ is solvable.

PF: L radical/K $\Longrightarrow$

$\exists\ K = K_0 \subset K_1 \subset \cdots \subset K_s = L$

where $K_i = K_{i-1}(\alpha_i)$

Let $n = lcm(n_i)$ s.t. $\alpha_i^{n_i} \in K_{i-1}$

Let ζ be prim. n^{th} rt of unity

$$\begin{array}{ccccccc} K = K_0 & \subset & K_1 & \subset & \cdots & \subset & K_s = L \\ \cap & & \cap & & & & \cap \\ K' = K_0(\zeta) & \subset & K_1' = K_1(\zeta) & \subset & \cdots & & K_s' = K_s(\zeta) = L' \end{array}$$

Now Lemmas 1 & 2 apply

$\rightsquigarrow Gal(L'/K) \supseteq Gal(L'/K_1') \supseteq Gal(L'/K_2') \supseteq \cdots \supseteq \{1\}$

These are normal subgps with

$Gal(L'/K_i')/Gal(L'/K_{i+1}') = Gal(K_{i+1}'/K_i')$ abelian

$\Longrightarrow Gal(L'/K)$ solvable

Fund thm of Galois theory

$$Gal(E/K) = Gal(L'/K)/Gal(L'/E)$$

$\Longrightarrow Gal(E/K)$ solvable.

$$\boxed{K \subset E \subset L'}$$

Overall goal:

If $f \in \mathbb{Q}[x]$ solvable by radicals, $\mathrm{Gal}(L/\mathbb{Q})$ solvable.

This means that splitting field L of $f(x)$ is contained in a radical extension L'

$$\rightsquigarrow \mathbb{Q} \subset \underset{\substack{\uparrow \\ \text{splitting field}}}{L} \subset \underset{\substack{\uparrow \\ \text{radical}}}{L'}$$

Don't know L' is normal

Lemma 4 • K char 0 field

• Let $f \in K[x]$ solvable by radicals

Then $\exists K \subset L'$ normal & radical containing splitting field L of $f(x)$

$$\rightsquigarrow \mathbb{Q} \subset L \subset L'$$

Pf: Know

• L splitting field of $f(x)$ normal (but not radical)

• $\exists L \subset L'$ with L' radical (but not nec. normal)

where $L = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_s = L'$

with $L_i = L_{i-1}(\alpha_i)$ $\alpha_i^{n_i} \in L_{i-1}$

We first handle a special case

We apply Lemma 5 inductively.

• Apply Lemma 5 to

$$K = E \subset L_1 \rightsquigarrow K \subset \underbrace{\overbrace{L_1 \subset L_1'}^{\text{radical}}}_{\text{normal}}$$

⋮

Lemma 5 $K \subset E \subset L = E(\alpha)$

char 0

Assume E normal$/K$

with $\alpha^n \in E$

Then $\exists\ K \subset E \subset L \subset L'$

with $E \subset L'$ radical &

$K \subset L'$ normal

Pf: E splitting field of some $f \in K[x]$

Let $p(x)$ be min. poly of α.

Take L' splitting field of

$f(x)\,p(x)$.

Goal: $f \in K[x]$ solvable by radicals

$\implies \mathrm{Gal}(L/K)$ solvable.

Here: L splitting field of f.

Pf: Lemma 4 $\implies$

$\exists\, K \subset L \subset L'$

radical & normal$/K$

Lemma 3 $\implies$

$\mathrm{Gal}(L/K)$ solvable. ✓